S.-T. Yau College Student Mathematics Contests 2026

Geometry and Topology

Problem 1. *Let $S \subset \mathbb{R}^3$ be a smooth regular surface without boundary, and let $p \in S$. Let $\{A_\varepsilon\}_{\varepsilon>0}$ be a family of regions in S such that each A_ε contains p, has area $|A_\varepsilon|$, and shrinks to $\{p\}$ as $\varepsilon \to 0$. Let $N : S \to \mathbb{S}^2$ be the **Gauss map**. Show that the Gaussian curvature at p satisfies*

$$|K(p)| = \lim_{\varepsilon \to 0} \frac{\text{Area}(N(A_\varepsilon))}{\text{Area}(A_\varepsilon)}.$$

Problem 2. *Let (M^2, g) be a smooth, compact, oriented two–dimensional Riemannian manifold with nonempty boundary ∂M. Assume that*

(1) the Gaussian curvature satisfies $K_g \geq 0$ on M, and
*(2) the geodesic curvature of the boundary satisfies $k_g \geq 1$ along ∂M. Here k_g denotes the geodesic curvature of ∂M, computed with respect to **the outward pointing unit normal** along the boundary.*

Suppose moreover that

$$\text{Length}(\partial M) \geq 2\pi.$$

Show that (M, g) is isometric to the unit disk $(\mathbb{D}^2(1), g_{\text{flat}})$ equipped with the standard Euclidean metric.

Problem 3. *Let G be a connected n–dimensional Lie group equipped with a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$.*

(1) Show that the sectional curvature of G is nonnegative.
(2) Assume that the Lie algebra $\mathfrak{g}$ of G has trivial center, i.e.,

$$\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\} = \{0\}.$$

Show that G is compact.

(3) Suppose that G is simply connected. Show that G decomposes as a direct product

$$G \cong G' \times \mathbb{R}^k,$$

where G' is a simply connected compact Lie group whose Lie algebra has trivial center, and $\mathbb{R}^k$ is the additive Lie group.

Problem 4. *Let $\mathbb{S}^{2026}$ be the unit sphere in $\mathbb{R}^{2027}$.*

(1) Compute the Euler class of the tangent bundle $T\mathbb{S}^{2026} \to \mathbb{S}^{2026}$ with $\mathbb{Z}_2$ coefficients.
(2) Let $U\mathbb{S}^{2026} \to \mathbb{S}^{2026}$ denote the unit tangent sphere bundle; that is, the fiber over each point is identified with $\mathbb{S}^{2025}$. Compute the Poincaré series

$$P_t(U\mathbb{S}^{2026}; \mathbb{Z}_2) = \sum_{i \geq 0} b_i(U\mathbb{S}^{2026}; \mathbb{Z}_2)\, t^i,$$

where $b_i(U\mathbb{S}^{2026}; \mathbb{Z}_2)$ denotes the i–th Betti number.

(3) *Let* $p, q \in \mathbb{S}^{2026}$ *be two non-antipodal points. Consider the space of piecewise smooth paths from* p *to* q*,*

$$\Omega(\mathbb{S}^{2026}; p, q) = \left\{ \gamma : [0,1] \to \mathbb{S}^{2026} \mid \gamma(0) = p,\ \gamma(1) = q \right\},$$

equipped with the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt.$$

(a) *Determine all critical points of* E*.*
(b) *The index* $\lambda(E, \gamma)$ *of* E *at a critical point* γ *is defined as the dimension of the maximal subspace of* $T_\gamma \Omega(\mathbb{S}^{2026}; p, q)$ *on which* $\operatorname{Hess} E$ *is negative definite. Compute* $\lambda(E, \gamma)$*.*
(c) *Determine the homotopy type of the based loop space* $\Omega(\mathbb{S}^{2026}, p)$*.*
(d) *Compute the Poincaré series of both the based loop space* $\Omega(\mathbb{S}^{2026}, p)$ *and the free loop space* $\Lambda \mathbb{S}^{2026}$ *with* $\mathbb{Z}_2$ *coefficients.*

Problem 5. *Let* M^n *be a connected, closed, smooth, aspherical manifold of dimension* $n \geq 1$*, that is, its universal covering space* $\widetilde{M}$ *is contractible.*

(1) *Prove that the universal cover* $\widetilde{M}$ *is noncompact.*
(2) *Prove that every nontrivial element of* $\pi_1(M)$ *has infinite order.*

Problem 6. *Let* (M^n, g) *be a complete Riemannian manifold of dimension* $n \geq 2$*. Suppose that there exists* $\epsilon > \frac{1}{4}$ *such that*

$$\operatorname{Ric}_g(x) \geq \epsilon \, (n-1) \, r(p,x)^{-2}$$

for all x *with* $r(p,x) = d(p,x)$ *sufficiently large.*

(1) *Show that* M *must be compact.*
(2) *Show, by example, that the conclusion in (1) fails if* $\epsilon \leq \frac{1}{4}$*.*